A NOTE ON AUTOMORPHISMS OF THE AFFINE CREMONA GROUP

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ABSTRACT. HANSPETER KRAFT and the author proved in [KS11] that every automorphism of the affine Cremona group $\mathcal{G}_n := \mathrm{Aut}(\mathbb{C}^n)$ is inner up to field automorphisms, when restricted to the subgroup of tame automorphisms $T\mathcal{G}_n$. We generalize this result in the following way: If an automorphism of the affine Cremona group $\mathcal{G}_3$ is the identity on the tame automorphisms $T\mathcal{G}_3$, then it also fixes the (non-tame) NAGATA automorphism.

0. **Introduction.** We denote throughout this note by $\mathcal{G}_n$ the group of polynomial automorphisms $\mathrm{Aut}(\mathbb{A}^n)$ of the complex affine space $\mathbb{A}^n = \mathbb{C}^n$. For polynomials $g_1, \ldots, g_n \in \mathbb{C}[x_1, \ldots, x_n]$, we use the notation $\mathbf{g} = (g_1, \ldots, g_n) \in \mathcal{G}_n$ to describe the automorphism

$$\mathbf{g}(a) = (g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)) \quad \text{for } a = (a_1, \ldots, a_n) \in \mathbb{A}^n .$$

The tame automorphism group $T\mathcal{G}_n$ is the subgroup of $\mathcal{G}_n$ generated by the affine linear automorphisms (i.e. the automorphisms $(g_1, \ldots, g_n)$ with $\deg(g_i) \leq 1$ for each i) and the triangular automorphisms (i.e. the automorphisms $(g_1, \ldots, g_n)$ where $g_i = g_i(x_i, \ldots, x_n)$ depends only on $x_i, \ldots x_n$ for each i). If D is a locally nilpotent derivation of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ (i.e. a $\mathbb{C}$-derivation such that for all $p \in \mathbb{C}[x_1, \ldots x_n]$ there exists $m \geq 0$ with $D^m(p) = 0$) we denote by $\exp(D)$ the automorphism $(e_1, \ldots, e_n) \in \mathcal{G}_n$ with

$$e_i = \sum_{k=0}^{\infty} \frac{1}{k!} D^k(x_i) \quad \text{for all } i = 0, \ldots, n .$$

The main result of [KS11] is the following

Theorem. *Let θ be an automorphism of $\mathcal{G}_n$. Then there exists $\mathbf{g} \in \mathcal{G}_n$ and a field automorphism $\tau \colon \mathbb{C} \to \mathbb{C}$ such that*

$$\theta(\mathbf{f}) = \tau(\mathbf{g} \circ \mathbf{f} \circ \mathbf{g}^{-1}) \quad \textit{for all tame automorphisms } \mathbf{f} \in T\mathcal{G}_n .$$

In dimension $n = 2$ all automorphisms are tame (cf. [Jun42] and [vdK53]). But in dimension $n = 3$, IVAN P. SHESTAKOV and UALBAI U. UMIRBAEV proved that the NAGATA automorphism $\mathbf{h} \in \mathcal{G}_3$ given by

$$\mathbf{h}(x, y, z) := (x + y(xz - \frac{1}{2}y^2) + \frac{1}{2}z(xz - \frac{1}{2}y^2)^2, y + z(xz - \frac{1}{2}y^2), z)$$

is non-tame (cf. [SU04]). A natural question is, whether the theorem above also extends to the NAGATA automorphism. More specifically, if θ is an automorphism

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of $\mathcal{G}_3$ such that $\theta|_{T\mathcal{G}_3} = \mathrm{id}$, does this imply $\theta(\mathbf{h}) = \mathbf{h}$? After some preparation we will give an affirmative answer to this question.

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1. **Preliminary results.** The Nagata automorphism $\mathbf{h}$ can be written as an exponential automorphism. Namely, $\mathbf{h} = \exp(pD)$ where

$$D = z\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} \quad \text{and} \quad p := xz - \frac{1}{2}y^2 \in \ker D\,.$$

Let $\mathbf{h}' = \exp(D)$. We denote by $\mathrm{Cent}(\mathbf{h}')$ the centralizer of $\mathbf{h}'$ in the group $\mathcal{G}_3$. Clearly, $\mathbf{h} \in \mathrm{Cent}(\mathbf{h}')$ and every automorphism of $\mathcal{G}_3$ fixes this centralizer. First, we want to describe $\mathrm{Cent}(\mathbf{h}')$, in order to deduce information about the restriction of an automorphism of $\mathcal{G}_3$ to $\mathrm{Cent}(\mathbf{h}')$. We denote by E the partial derivative with respect to x. The subgroups of $\mathcal{G}_3$ listed below are clearly contained in $\mathrm{Cent}(\mathbf{h}')$.

$$\begin{aligned} C &:= \{\,(\alpha x, \alpha y, \alpha z) \mid \alpha \in \mathbb{C}^*\,\} \\ F_1 &:= \{\,\exp(qD) \mid q \in \ker D\,\} \\ F_2 &:= \{\,\exp(hE) \mid h \in \ker E \cap \ker D\,\} \end{aligned}$$

Proposition. *We have a semi-direct product decomposition*

$$\mathrm{Cent}(\mathbf{h}') = C \ltimes (F_2 \ltimes F_1)\,.$$

Proof. The kernel $R := \ker D$ is the polynomial ring $\mathbb{C}[z,p]$, according to Proposition 2.3 [DF98]. Clearly, we have $R[x] = \mathbb{C}[z,x,y^2]$ and hence a decomposition $\mathbb{C}[x,y,z] = R[x] \oplus yR[x]$. Let $\mathbf{f} = (f_1,f_2,f_3) \in \mathrm{Cent}(\mathbf{h}')$. Now, we can write $f_1 = v + yq$ for polynomials $v, q \in R[x]$. By definition, in $\mathbb{C}[x,y,z,t]$ we have

$$v(x+ty+\frac{1}{2}t^2z) + (y+tz)q(x+ty+\frac{1}{2}t^2z) \;=\; v(x) + yq(x) + tf_2 + \frac{1}{2}t^2 f_3\,.$$

A comparison of the coefficients with respect to the variable t shows that $v = r + sx$ with $r, s \in R$ and $q \in R$. Hence, we have $f_1 = r + sx + qy,\ f_2 = sy + qz,\ f_3 = sz$. Since $\mathbf{f}$ is an automorphism, $s \in \mathbb{C}^*$. Up to post composition with an element of C we can assume that $s = 1$. Thus,

$$\mathbf{f} \circ \exp(qD)^{-1} = (x + r - \frac{1}{2}q^2 z, y, z)\,.$$

One can see that this automorphism lies in F_2. Hence, the proposition follows. $\square$

Let $K \subseteq \mathrm{Cent}(\mathbf{h}')$ be the subgroup $\{\,\exp(qD) \mid q \in p\mathbb{C}[pz^2]\,\}$, and let $H := C \ltimes K \subseteq \mathrm{Cent}(\mathbf{h}')$. One can see that H consists of all automorphisms in $\mathrm{Cent}(\mathbf{h}')$ that commute with the group $\{\,(\alpha^3 x, \alpha y, \alpha^{-1} z) \mid \alpha \in \mathbb{C}^*\,\}$. This implies that H is preserved under all automorphisms of $\mathcal{G}_3$ that are the identity on $T\mathcal{G}_3$. The torus $T := \{\,(\beta^2\gamma^{-1}x, \beta y, \gamma z) \mid \beta, \gamma \in \mathbb{C}^*\,\}$ acts on H by conjugation. A subgroup $U \subseteq H$ is one-dimensional and unipotent if it is of the form

$$U = \{\,\exp(qtD) \mid t \in \mathbb{C}\,\} \quad \text{for some } q \in p\mathbb{C}[pz^2]\,.$$

Such a U admits in a natural way the structure of a one-dimensional $\mathbb{C}$-vector space. In the next lemma we study the one-dimensional unipotent subgroups of H that are normalized by T.

Lemma. *Let $U \subseteq H$ be a one-dimensional unipotent subgroup normalized by T. Then there exists $k \geq 0$ such that $\exp(p(pz^2)^k D) \in U$ and T acts by conjugation on U via the character*

$$\lambda_k \colon T \to \mathbb{C}^*, \quad (\beta^2\gamma^{-1}, \beta, \gamma) \mapsto (\beta\gamma)^{2k+1}.$$

Proof. As U is unipotent we have $U \subseteq K$. Hence, $\exp(qD) \in U$ for some $q \in p\mathbb{C}[pz^2]$. Since U is normalized by T it follows that $q = p(pz^2)^k$ for some $k \geq 0$. Now, one can see that T acts on U via the claimed character λ_k. □

2. The main theorem.

After these preliminary results, we are now able to state and prove the main result of this note.

Theorem. *Let $\theta : \mathcal{G}_3 \to \mathcal{G}_3$ be an automorphism of abstract groups that is the identity on $T\mathcal{G}_3$. Then $\theta(\mathbf{h}) = \mathbf{h}$.*

Proof. Let $U \subseteq H$ be the one-dimensional unipotent subgroup with $\mathbf{h} = \exp(pD) \in U$. We claim that $\theta(U) = U$ and the restriction of θ to U is $\mathbb{C}$-linear. As in the proof of Proposition 6.1(b) [KS11] it follows that $U' := \theta(U) \subseteq H$ is a one-dimensional unipotent subgroup normalized by T (since $\theta|_T = id_T$). According to the lemma above there exist $k, k' \geq 0$ such that T acts on U via the character λ_k and T acts on U' via the character $\lambda_{k'}$. Since

$$(*) \qquad \theta(\lambda_k(\mathbf{t}) \cdot \mathbf{u}) = \theta(\mathbf{t} \circ \mathbf{u} \circ \mathbf{t}^{-1}) = \lambda_{k'}(\mathbf{t}) \cdot \theta(\mathbf{u}) \quad \text{for } \mathbf{t} \in T, \ \mathbf{u} \in U$$

it follows that λ_k and $\lambda_{k'}$ have the same kernel (cf. proof of Proposition 7.1(b) [KS11]). Since $k, k' \geq 0$ this implies that $k = k'$. Therefore we have $U' = U$ and θ is $\mathbb{C}$-linear on U according to $(*)$.

We claim that $\theta|_U = \text{id}$. As we have already proved, there exists $a \in \mathbb{C}^*$ such that $\theta(\mathbf{u}) = a \cdot \mathbf{u}$ for all $\mathbf{u} \in U$. We have

$$(x-1, y, z) \circ \exp(pD) \circ (x+1, y, z) = \exp((p+z)D) = \exp(pD) \circ \exp(zD).$$

Applying θ to the last equation and using the fact that $\exp(zD)$ and $(x \pm 1, y, z)$ are tame automorphisms yields $\exp(a(p+z)D) = \exp(apD) \circ \exp(zD)$. Thus, it follows that $\exp(azD) = \exp(zD)$, and hence we have $a = 1$, proving the claim. Since $\mathbf{h} \in U$ this finishes the proof of the theorem. □

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